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A variant of the Banach-Mazur game and knot points of typical continuous functions

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1 Knot points of typical continuous functions

We begin by giving the statement of the main theorem of the author's PhD thesis [Sa1], established jointly with David Preiss. For the background and historical remarks, see [Sa1].

We write I for the unit interval $[0,1]$ and $C(I)$ for the set of all real-valued continuous functions defined on I.

Definition 1.1.
Let $f \in C(I)$. A point $a \in I$ is called a **knot point** of f if

$$\limsup_{x \downarrow a} \frac{f(x)-f(a)}{x-a} = \limsup_{x \uparrow a} \frac{f(x)-f(a)}{x-a} = \infty,$$
$$\liminf_{x \downarrow a} \frac{f(x)-f(a)}{x-a} = \liminf_{x \uparrow a} \frac{f(x)-f(a)}{x-a} = -\infty.$$

Here if a is an endpoint of the interval I, then we ignore the two undefined limits. We denote by $N(f)$ the set of all points in I that are *not* knot points of f.

If $f \in C(I)$ is differentiable, then f has no knot points, so $N(f) = I$. However most $f \in C(I)$ are so bad that $N(f)$ is fairly small. To make this statement precise, we introduce the term *typical*. We give $C(I)$ the topology induced by the supremum norm.

Definition 1.2.
We say that a **typical** (**generic**) $f \in C(I)$ has property P if the set of all $f \in C(I)$ with property P is residual in $C(I)$.

Recall that a subset A of a topological space is said to be **nowhere dense** if the closure of A has empty interior; A is **meagre** (**first category**) if A can be expressed as a countable union of nowhere dense sets; A is **residual** (**comeagre**) if its complement A^c is meagre.

We shall characterise those families $\mathcal{F}$ of subsets of I for which $N(f) \in \mathcal{F}$ for a typical $f \in C(I)$. Since $N(f)$ is always an F_σ subset of I, we may assume that $\mathcal{F}$ is a subfamily of $\mathcal{F}_\sigma$, the family of all F_σ subsets of I. For a subfamily $\mathcal{F}$ of $\mathcal{F}_\sigma$, our main theorem asserts that $N(f) \in \mathcal{F}$ for a typical $f \in C(I)$ if and only if $\mathcal{F}$ is *large*.

To define what it means for $\mathcal{F}$ to be large, we write $\mathcal{K}$ for the family of all closed subsets of I, and equip $\mathcal{K}$ with the Hausdorff metric d. Recall that, writing $B(x,r)$ for the open ball of centre x and radius r, we define the Hausdorff metric by

$$d(K,L) = \inf\left\{r > 0 \,\middle|\, \bigcup_{x\in K} B(x,r) \supset L,\ \bigcup_{x\in L} B(x,r) \supset K\right\}$$

for nonempty $K, L \in \mathcal{K}$, and $d(K,\emptyset) = 1$ for $K \in \mathcal{K} \setminus \{\emptyset\}$. Its countable product $\mathcal{K}^{\mathbb{N}}$ is furnished with the product topology.

Definition 1.3 ([Sa2, Definition 1.2]).
A subfamily $\mathcal{F}$ of $\mathcal{F}_\sigma$ is said to be **residual** if $\{(K_n) \in \mathcal{K}^{\mathbb{N}} \mid \bigcup_{n=1}^{\infty} K_n \in \mathcal{F}\}$ is a residual subset of $\mathcal{K}^{\mathbb{N}}$.

Theorem 1.4 ([Sa1, Main Theorem]).
A subfamily $\mathcal{F}$ of $\mathcal{F}_\sigma$ is residual if and only if $N(f) \in \mathcal{F}$ for a typical $f \in C(I)$.

2 A variant of the Banach-Mazur game

A complete proof of Theorem 1.4 can be found in [Sa1]. An important ingredient of the proof there is to rephrase residuality in terms of the Banach-Mazur game.

Definition 2.1.
For a topological space X and its subset S, the **(X,S)-Banach-Mazur game** is described as follows. Players I and II alternately choose a nonempty open subset of X:

$$\begin{array}{llllll} \text{I:} & U_1 & & U_2 & & \\ \text{II:} & & V_1 & & V_2 & \cdots \end{array}$$

where U_m and V_m are nonempty open subsets of X for all $m \in \mathbb{N}$, with the restriction that V_m must be contained in U_m for every $m \in \mathbb{N}$ and U_m must be contained in V_{m-1} for every $m \in \mathbb{N} \setminus \{1\}$. Player II wins if $\bigcap_{m=1}^{\infty} V_m \subset S$; otherwise Player I wins.

Theorem 2.2 ([Ox]).
In the (X,S)-Banach-Mazur game, Player II has a winning strategy if and only if S is residual in X.

In [Sa1] we first use the Banach-Mazur game to prove that if $\mathcal{F}$ is residual, then $N(f) \in \mathcal{F}$ for a typical $f \in C(I)$; then we invoke results in descriptive set theory to show the converse. In order to make the descriptive set-theoretical results applicable, we have to prove a slightly stronger statement than the first implication. However the first implication itself can be proved in a simpler manner by using a variant of the Banach-Mazur game. Unfortunately the simpler proof is still too complicated to be included here, so what we shall do below is to detail the variant of the Banach-Mazur game used there.

We first introduce an equivalent variant of the Banach-Mazur game:

Proposition 2.3.

Let X be a topological space, S a subset of X, and $\mathcal{A}$ a family of pairs of a point of X and its open neighbourhood. Suppose that for every nonempty open subset O of X there exists $(x, U) \in \mathcal{A}$ with $U \subset O$. We consider the following game. Players I and II alternately choose an element of $\mathcal{A}$:

$$\begin{array}{llllll} \text{I:} & (x_1, U_1) & & (x_2, U_2) & & \\ \text{II:} & & (y_1, V_1) & & (y_2, V_2) & \cdots \end{array}$$

where $(x_m, U_m), (y_m, V_m) \in \mathcal{A}$ for all $m \in \mathbb{N}$, with the restriction that y_m must belong to U_m for every $m \in \mathbb{N}$ and x_m must belong to V_{m-1} for every $m \in \mathbb{N} \setminus \{1\}$. Player II wins if $\bigcap_{m=1}^{\infty} V_m \subset S$; otherwise Player I wins.

Then Player II has a winning strategy in this game if and only if S is residual in X.

Proof.

Suppose first that S is residual in X. Then Player II has a winning strategy in the (X, S)-Banach-Mazur game by Theorem 2.2. Using the winning strategy in the Banach-Mazur game, Player II can obtain a winning strategy in our game in the following manner:

$$\begin{array}{lccc} & \text{our game} & & \text{Banach-Mazur game} \\ \text{I:} & (x_1, U_1) & \longrightarrow & \tilde{U}_1 \\ \text{II:} & (y_1, V_1) & \longleftarrow & \tilde{V}_1 \\ \text{I:} & (x_2, U_2) & \longrightarrow & \tilde{U}_2 \\ \text{II:} & (y_2, V_2) & \longleftarrow & \tilde{V}_2 \\ & \vdots & & \vdots \end{array}$$

Broadly speaking, given the mth move (x_m, U_m) of Player I in our game, Player II transfers it to the Banach-Mazur game to obtain the mth imaginary move $\tilde{U}_m$ of Player I, and then transfers to our game the imaginary reply $\tilde{V}_m$ given by the winning strategy to get her real reply (y_m, V_m). The details of the transfers are as follows:

$\tilde{U}_m = U_m \cap V_{m-1}$ ($\tilde{U}_m = U_m$ if $m = 1$), and (y_m, V_m) is an element of $\mathcal{A}$ such that $V_m \subset \tilde{V}_m$. Note that this procedure gives legal moves.

Obeying this method, Player II can win because $\bigcap_{m=1}^{\infty} V_m \subset \bigcap_{m=1}^{\infty} \tilde{V}_m \subset S$, where the latter inclusion follows from the fact that the sets $\tilde{V}_m$ were given by the winning strategy in the Banach-Mazur game.

The converse can be proved in the same way. ∎

The residuality of subfamilies of $\mathcal{F}_\sigma$ has been defined via the space $\mathcal{K}^{\mathbb{N}}$ in Definition 1.3, but the subspace $\mathcal{K}^{\mathbb{N}}_{\nearrow}$ of increasing sequences gives an equally natural definition:

Definition 2.4 ([Sa2, Definition 1.2]).
Let $\mathcal{K}^{\mathbb{N}}_{\nearrow}$ denote the set of all increasing sequences in $\mathcal{K}^{\mathbb{N}}$:

$$\mathcal{K}^{\mathbb{N}}_{\nearrow} = \{(K_n) \in \mathcal{K}^{\mathbb{N}} \mid K_1 \subset K_2 \subset \cdots\},$$

equipped with the relative topology. A subfamily $\mathcal{F}$ of $\mathcal{F}_\sigma$ is said to be **$\nearrow$-residual** if $\{(K_n) \in \mathcal{K}^{\mathbb{N}}_{\nearrow} \mid \bigcup_{n=1}^{\infty} K_n \in \mathcal{F}\}$ is a residual subset of $\mathcal{K}^{\mathbb{N}}_{\nearrow}$.

It is shown in [Sa2] that the two definitions of residuality are equivalent.

Definition 2.5.
For $N \in \mathbb{N}$ and $t > 0$, we say that $(K_n) \in \mathcal{K}^{\mathbb{N}}$ is **(N,t)-close** (resp. **(N,t)-$\nearrow$-close**) to $(L_n) \in \mathcal{K}^{\mathbb{N}}$ if $d(K_n, L_n) < t$ (resp. $d(\bigcup_{j=1}^{n} K_j, \bigcup_{j=1}^{n} L_j) < t$) for $n = 1, \ldots, N$.

Remark 2.6.
The (N,t)-closeness implies the (N,t)-$\nearrow$-closeness, but the converse is not true in general.

Definition 2.7.
For a subfamily $\mathcal{F}$ of $\mathcal{F}_\sigma$, we define three games called the **disjoint game**, the **monotone game**, and the **mixed game**.

Let $\mathcal{D}$ denote the set of all sequences whose terms are pairwise disjoint finite subsets of I. In any of these games, Players I and II alternately choose a sequence in $\mathcal{D}$, a positive integer, and a positive real number:

I:	$(K_n^{(1)}), a^{(1)}, r^{(1)}$		$(K_n^{(2)}), a^{(2)}, r^{(2)}$		$\cdots$
II:		$(L_n^{(1)}), b^{(1)}, s^{(1)}$		$(L_n^{(2)}), b^{(2)}, s^{(2)}$	

where $(K_n^{(m)}), (L_n^{(m)}) \in \mathcal{D}$, $a^{(m)}, b^{(m)} \in \mathbb{N}$, and $r^{(m)}, s^{(m)} > 0$ for all $m \in \mathbb{N}$.

(1) In the disjoint game, $(L_n^{(m)})$ must be $(a^{(m)}, r^{(m)})$-close to $(K_n^{(m)})$ for every $m \in \mathbb{N}$ and $(K_n^{(m)})$ must be $(b^{(m-1)}, s^{(m-1)})$-close to $(L_n^{(m-1)})$ for every $m \in \mathbb{N} \setminus \{1\}$. Player II wins if $\bigcup_{n=1}^{\infty} K_n \in \mathcal{F}$ whenever $(K_n) \in \mathcal{K}^{\mathbb{N}}$ is $(b^{(m)}, s^{(m)})$-close to $(L_n^{(m)})$ for all $m \in \mathbb{N}$; otherwise Player I wins.

(2) In the monotone game, $(L_n^{(m)})$ must be $(a^{(m)}, r^{(m)})$-$\nearrow$-close to $(K_n^{(m)})$ for every $m \in \mathbb{N}$ and $(K_n^{(m)})$ must be $(b^{(m-1)}, s^{(m-1)})$-$\nearrow$-close to $(L_n^{(m-1)})$ for every $m \in \mathbb{N} \setminus \{1\}$. Player II wins if $\bigcup_{n=1}^{\infty} K_n \in \mathcal{F}$ whenever $(K_n) \in \mathcal{K}^{\mathbb{N}}_{\nearrow}$ is $(b^{(m)}, s^{(m)})$-$\nearrow$-close to $(L_n^{(m)})$ for all $m \in \mathbb{N}$; otherwise Player I wins.

(3) In the mixed game, $(L_n^{(m)})$ must be $(a^{(m)}, r^{(m)})$-close to $(K_n^{(m)})$ for every $m \in \mathbb{N}$ and $(K_n^{(m)})$ must be $(b^{(m-1)}, s^{(m-1)})$-$\nearrow$-close to $(L_n^{(m-1)})$ for every $m \in \mathbb{N} \setminus \{1\}$. Player II wins if $\bigcup_{n=1}^{\infty} K_n \in \mathcal{F}$ whenever $(K_n) \in \mathcal{K}^{\mathbb{N}}_{\nearrow}$ is $(b^{(m)}, s^{(m)})$-$\nearrow$-close to $(L_n^{(m)})$ for all $m \in \mathbb{N}$; otherwise Player I wins.

The set $\mathcal{D}$ defined above is dense in $\mathcal{K}^{\mathbb{N}}$, and the set $\{(\bigcup_{j=1}^{n} K_j) \in \mathcal{K}^{\mathbb{N}}_{\nearrow} \mid (K_n) \in \mathcal{D}\}$ is dense in $\mathcal{K}^{\mathbb{N}}_{\nearrow}$.

Proposition 2.8.

For a subfamily $\mathcal{F}$ of $\mathcal{F}_\sigma$, the following conditions are equivalent:

(1) *Player II has a winning strategy in the disjoint game for $\mathcal{F}$;*

(1a) *$\mathcal{F}$ is residual;*

(2) *Player II has a winning strategy in the monotone game for $\mathcal{F}$;*

(2a) *$\mathcal{F}$ is $\nearrow$-residual;*

(3) *Player II has a winning strategy in the mixed game for $\mathcal{F}$.*

Outline Proof.

Proposition 2.3 shows that (1) is equivalent to (1a) and that (2) is equivalent to (2a). It is easy to see that Remark 2.6 ensures that (3) implies both (1) and (2). It is proved in [Sa2] that (1) and (2) are equivalent, and in fact the proof there shows that each of (1) and (2) implies (3). ∎

The mixed game allows us to prove the following propositions, which is equivalent to saying that if $\mathcal{F}$ is residual, then $N(f) \in \mathcal{F}$ for a typical $f \in C(I)$:

Proposition 2.9.

Let $\mathcal{F}$ be a subfamily of $\mathcal{F}_\sigma$ for which Player II has a winning strategy in the mixed game. Then Player II has a winning strategy in the $(C(I), S)$-Banach Mazur game, where $S = \{f \in C(I) \mid N(f) \in \mathcal{F}\}$.

Even the proof of this proposition is so complicated that we shall not go into further details here.

References

[Ox] J. C. Oxtoby, *The Banach-Mazur game and Banach category theorem*, Contributions to the theory of games, vol. 3, Ann. Math. Stud. **39** (1957), 159–163.

[Sa1] S. Saito, *Knot points of typical continuous functions and Baire category in families of sets of the first class*, PhD thesis submitted to the University of London, available on the author's website: `http://www2.math.kyushu-u.ac.jp/~ssaito/eng/maths/thesis.pdf`.

[Sa2] S. Saito, *Residuality of families of $\mathcal{F}_\sigma$ sets*, Real Anal. Exch. **31** (2005/2006), no. 2, 477–487

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